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Uniform Modal Damping of Rings by an Extended Node Control Theorem

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Introduction

In principle, a control system can dampen any combination of the modes of a vibrating structure. In certain applications, one wants to preserve the uncontrolled mode shapes and resonant frequencies of the lower modes. When point control forces and sensors are used, the placement of the control forces and sensors on the structure plays an important role in achieving these objectives. To this end, the natural control theory provides a general strategy for designing a control system that preserves the resonant frequencies and mode shapes of the structure while uniformly damping the modes.[1–3] Natural control is an application of the independent modal space control method due to Meirovitch and Baruh.[4] Weaver and Silverberg[5] applied natural control to uniform beams and developed a control force and sensor placement strategy called node control. This strategy is based on the node control theorem, which states: "If a uniform beam with homogeneous boundary conditions and with the lowest N modes participating in the system response is subject to N direct state feedback control forces placed at the N nodes of the $(N+1)$th mode, then the control gains can be selected such that the following properties apply to the controlled system: 1) frequency invariance; 2) mode invariance; 3) uniform damping."[5]

This Note extends the node control theorem for uniform beams to uniform rings. In contrast with uniform beams, uniform rings possess no boundary conditions and they are degenerate in the sense that, for each natural frequency, there are two orthogonal mode shapes, i.e., a sine mode and a cosine mode. In this Note we shall show that $2M+2$ control forces evenly spaced around the ring can uniformly dampen the lowest $2M+1$ modes and preserve the associated natural frequencies and mode shapes.

Elastic Ring Equations

The equations of motion for a uniform ring can be written as[5]

$$\rho A\ddot{v} - \left(\frac{EA}{R^2} + \frac{EI}{R^2}\right)\frac{\partial^2 v}{\partial\theta^2} - \frac{EA}{R^2}\frac{\partial w}{\partial\theta} + \frac{EI}{R^4}\frac{\partial^3 w}{\partial\theta^3} = 0$$
$$\rho A\ddot{w} - \frac{EA}{R^2}\frac{\partial v}{\partial\theta} - \frac{EI}{R^4}\frac{\partial^3 v}{\partial\theta^3} + \frac{EA}{R^2}w + \frac{EI}{R^4}\frac{\partial^4 w}{\partial\theta^4} = r + u \tag{1}$$

Received June 16, 1993; revision received March 29, 1994; accepted for publication June 6, 1994.

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where ρ is the mass density of the ring, E is Young's modulus, A is the cross-sectional area, R is the mean radius, w is the normal displacement of the ring, and v is the in-plane motion. Both w and v are functions of location θ and time t. The excitation in the normal direction is r, and u is the normal control action. Both r and u are functions of θ and t. It is assumed that there is no in-plane external force so that the in-plane inertial term is negligible. This assumption simplifies the analysis since the first equation of (1) becomes time independent. However, the assumption has little effect on the natural frequencies of the ring.

Since the ring is uniform and continuous, the above equations are subject to no boundary conditions. Instead, we impose the conditions that w, v, r, and u be periodic functions of θ. The modal expansions of these quantities are given by

$$w(\theta,t) = \sum_{n=-M}^{M} w_n(t)e^{jn\theta}, \qquad v(\theta,t) = \sum_{n=-M}^{M} v_n(t)e^{jn\theta}$$
$$u(\theta,t) = \sum_{n=-M}^{M} u_n(t)e^{jn\theta}, \qquad r(\theta,t) = \sum_{n=-M}^{M} r_n(t)e^{jn\theta} \tag{2}$$

where $j^2=-1$, w_n and v_n are modal displacements, r_n is a modal excitation, and u_n is a modal control force. The expansions are truncated at $n=\pm M$. Note that in order for the expansions of Eq. (3) to be real valued, the coefficients corresponding to n must be the complex conjugates of those corresponding to $-n$. Each term in the expansion of $w(\theta,t)$ represents a wave traveling around the ring and will be loosely referred to as a complex mode, or simply a mode of the ring. The sum of the $+n$th mode and the $-n$th mode results in a real deformation pattern consisting of a sine mode and a cosine mode. The number of nodes of the real deformation pattern is equal to $2n$. The nodes are evenly spaced along the ring and may move around the ring as the phase of the sine and cosine modes changes.

Substituting the expansions of Eqs. (2) into the equations of motion (1) leads to a set of decoupled equations for the coefficients w_n and v_n. The first equation of (1) yields an algebraic relationship between v_n and w_n, which allows the elimination of v_n from the second equation of (1), resulting in a set of second-order ordinary differential equations:

$$\ddot{w}_n + \omega_n^2 w_n = \hat{r}_n + \hat{u}_n, \qquad -M \le n \le M \tag{3}$$

where $\hat{r}_n = r_n/\rho A$ and $\hat{u}_n = u_n/\rho A$ are the normalized excitation and control modal forces and ω_n is the open-loop natural frequency of the nth mode given by

$$\omega_n^2 = \begin{cases} \dfrac{E}{\rho R^2}, & n = 0 \\ \dfrac{1}{\rho A}\left(\dfrac{EA}{R^2} + \dfrac{EI}{R^4}n^4 - \dfrac{[EA/R^2 + (EI/R^4)n^2]^2}{EA/R^2 + EI/R^4}\right) & n \ne 0 \end{cases} \tag{4}$$

Control System

Consider N discrete point control forces expressed as

$$u(\theta,t) = \sum_{i=1}^{N} f_i(t)\,\delta(\theta-\theta_i) = \sum_{i=1}^{N}[h_i\dot{w}(\theta_i,t) + g_i w(\theta_i,t)]\,\delta(\theta-\theta_i) \tag{5}$$

where h_i is the velocity feedback gain and g_i is the position feedback gain of the ith control force $f_i(t)$ located at θ_i. Then,

$$\hat{u}_n = \sum_{m=-M}^{M} H_{nm}\dot{w}_m + \sum_{m=-M}^{M} G_{nm}w_m \tag{6}$$

where G_{nm} and H_{nm} are control gain matrices given by

$$H_{nm} = \frac{1}{2\pi\rho A}\sum_{i=1}^{N} h_i e^{j(m-n)\theta_i}, \qquad G_{nm} = \frac{1}{2\pi\rho A}\sum_{i=1}^{N} g_i e^{j(m-n)\theta_i} \tag{7}$$

Substituting Eq. (6) into Eq. (3), we obtain the modal equations of the closed-loop system

$$\ddot{w}_n + \omega_n^2 w_n = \sum_{m=-M}^{M} H_{nm}\dot{w}_m + \sum_{m=-M}^{M} G_{nm} w_m, \qquad -M \le n \le M \tag{8}$$

Note that the modal excitation $\hat{r}_n$ is neglected in Eq. (8). Equation (8) represents a set of coupled differential equations. Assuming a solution to Eq. (8) in the form

$$w_n(t) = W_n e^{\lambda t} \tag{9}$$

we obtain the closed-loop eigenvalue problem

$$\sum_{m=-M}^{M} \left[\left(\lambda^2 + \omega_n^2\right)\delta_{nm} - \lambda H_{nm} - G_{nm}\right] W_m = 0, \qquad -M \le n \le M \tag{10}$$

where λ is the closed-loop eigenvalue, W_n is the closed-loop eigenvector, and δ_{nm} is the Kronecker delta.

As stated earlier, we are interested in a control system that preserves open-loop mode shapes and open-loop natural frequencies and dampens the modes at a uniform rate. To meet the first objective, the closed-loop system, like the open-loop system, has to be decoupled. This requires the matrices G_{nm} and H_{nm} to be diagonal. Thus, we obtain the decoupled eigenvalue problem

$$\left(\lambda_n^2 + \omega_n^2 - \lambda_n H_{nn} - G_{nn}\right) W_n = 0, \qquad -M \le n \le M \tag{11}$$

where λ_n is a complex eigenvalue associated with the nth mode and can be written as

$$\lambda_n = -\alpha_n \pm j\beta_n \tag{12}$$

where α_n is the decay rate of the nth mode and β_n is the closed-loop frequency. To preserve the open-loop frequencies, we require that $\omega_n = \beta_n$. This leads to

$$H_{nn} = -2\alpha_n, \qquad G_{nn} = -\alpha_n^2 \tag{13}$$

The objective of uniform damping is readily achieved by letting $\alpha_n = \alpha$ for all n.

In this study, we shall only consider a control design with equal gains for all control forces ($h_i = h$ and $g_i = g$). To determine the locations θ_i and the number of control forces N that can accomplish the above natural control design, we turn to the node control theorem for uniform beams[5] and examine the nodes of elastic ring modes.

As discussed earlier, the nodes of the real deformation pattern resulting from the $\pm n$th modes are evenly spaced around the ring. This suggests that, if we want to control the lower order modes up to $n = \pm M$, we should first try to evenly place $2(M+1)$ control forces on the ring because the real deformation pattern resulting from the modes $n = \pm(M+1)$ has $2(M+1)$ evenly spaced nodes. The location of the control forces can be expressed as

$$\theta_i = \frac{2\pi(i-1)}{N} + \gamma_0, \qquad i = 1, 2, \ldots, N \tag{14}$$

where $N = 2M + 2$ and γ_0 is an arbitrary reference angle. Note that θ_i given in Eq. (14) is not necessarily at a node of the real deformation pattern resulting from the modes $n = \pm(M+1)$ as the position of these nodes remains arbitrary, although they must be evenly spaced.

For example, for $M = 1$, we need four control forces evenly spaced around the ring. Thus, we have

$$H = \frac{3}{2\pi\rho A}\begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix}, \qquad G = \frac{3}{2\pi\rho A}\begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} \tag{15}$$

where

$$h_i = \frac{-4\pi\rho A\alpha}{N}, \qquad g_i = \frac{-2\pi\rho A\alpha^2}{N} \tag{16}$$

More generally, the matrices G and H can be expressed as

$$G_{nm} = -\alpha^2 F(n,m), \qquad H_{nm} = -2\alpha F(n,m) \tag{17}$$

where $F(n,m)$ is defined as

$$F(n,m) = \begin{cases} 1, & n = m \\ 0, & n \ne m \text{ and } |n-m| \ne kN, k = 1, 2, \ldots \\ 1, & |n-m| = kN, k = 1, 2, \ldots \end{cases} \tag{18}$$

Equation (18) suggests that G and H are diagonal when n and m are within the range of the controlled modes and are not diagonal when n and m are beyond the range of the controlled modes. These nondiagonal elements of G and H cause the control spillover to the higher order modes. Consider the case when $M = 1$. The controlled modes are $n = -1, 0, 1$. Truncating the expansion of $w(\theta, t)$ at $n = \pm 3$, we obtain

$$H = \begin{bmatrix} -2\alpha & 0 & 0 & 0 & -2\alpha & 0 & 0 \\ 0 & -2\alpha & 0 & 0 & 0 & -2\alpha & 0 \\ 0 & 0 & -2\alpha & 0 & 0 & 0 & -2\alpha \\ 0 & 0 & 0 & -2\alpha & 0 & 0 & 0 \\ -2\alpha & 0 & 0 & 0 & -2\alpha & 0 & 0 \\ 0 & -2\alpha & 0 & 0 & 0 & -2\alpha & 0 \\ 0 & 0 & -2\alpha & 0 & 0 & 0 & -2\alpha \end{bmatrix}$$

$$G = (\alpha/2)H \tag{19}$$

The inner 3×3 matrix corresponding to the controlled modes from $n = -1$ to $n = 1$ is diagonal, as expected. The modes $n = \pm 2$ are coupled to one another but are not coupled to any of the lower order modes. Thus there is no control spillover to the modes $n = \pm 2$. Beyond $n = \pm 2$, there is coupling of higher order modes to the controlled modes, causing control spillover. In general, only the odd- (even-) order modes couple to the odd (even) controlled modes.

Numerical Results

In this section we demonstrate the effects of control spillover via numerical simulations. In the simulations, six evenly spaced control forces beginning at an angle $\gamma_0 = \frac{1}{4}\pi$ were used to control the modes from $n = -2$ to $n = 2$. The modes from $n = -4$ to $n = 4$ are included in the expansion. The decay rate is chosen to be $\alpha = 100$. The cosine part of the mode $n = 2$ is given a unit initial displacement. Mode $n = 2$ begins with the unit initial displacement and decays at a rate of $\alpha = 100$. Because of the control spillover, mode $n = 4$ initially at rest builds up some finite response and then is dampened out quickly by the control system. This behavior is illustrated by Fig. 1a. This demonstrates two key properties of the control spillover:

1) If the decay rate α is low, the effect of the control spillover is small.

2) The modes that are excited by control spillover are dampened out along with the controlled modes.

Figure 1b shows the effect of the control system on modes $n = \pm 3$ one order above the highest controlled mode. Note that the real deformation pattern resulting from modes $n = \pm 3$ has six evenly

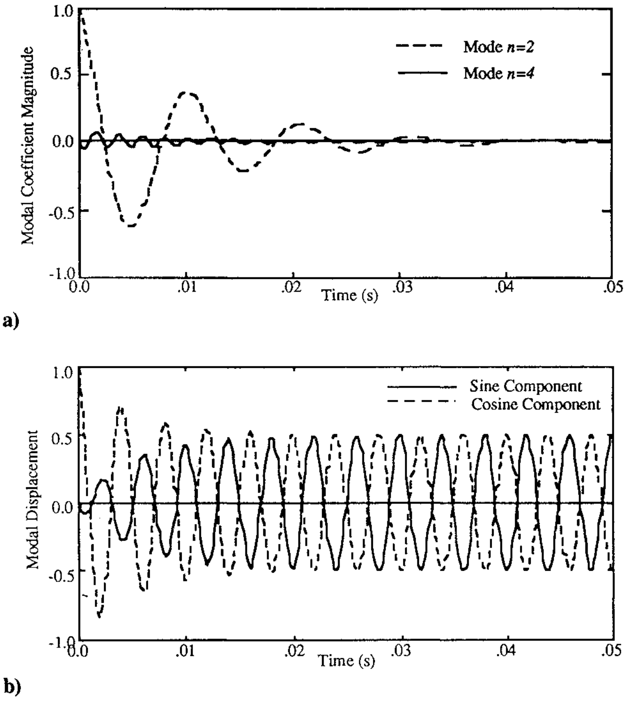

Fig. 1 Effect of control spillover ($M = 2, -4 \leq n \leq 4, \alpha = 100$): a) transient response of modes $n = 2$, and $n = 4$ and b) transient response of modes $n = -3$, and $n = 3$.

spaced nodes on the ring. Initially, the cosine part of mode $n = 3$ is excited. It then decays as the controller shifts some of its energy into the sine part. After about 0.015 s, the magnitudes of the sine part and the cosine part are about the same and there is little change from that point on. This shows that the control system has shifted the nodes of the real deformation pattern resulting from modes $n = \pm 3$ so that they lie very near the control forces. Hence, the control forces can no longer affect modes $n = \pm 3$. This phenomenon is referred to as mode rotation.

Conclusions

We have presented a method for uniform modal damping of the $2M + 1$ lowest modes of elastic rings by using velocity and position feedback of $2(M + 1)$ control forces evenly spaced around the ring. The method makes use of natural control and extends the node control theorem for uniform beams to uniform rings. The control system developed in this Note has been shown to provide uniform damping to the controlled modes of the ring and to preserve the mode shapes and natural frequencies of the controlled modes. The effects of control spillover have been studied in numerical simulations and the phenomena of mode rotation has been discussed.

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Alternative Variable Transformation for Simulation of Multibody Dynamic Systems

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I. Introduction

THE study of multibody dynamics (MBD) problems has emerged as an important area of research pertaining to the dynamics and control of spacecraft. The MBD system is usually modeled by a set of differential-algebraic equations, including a Jacobian matrix characterizing the kinematic constraints embedded in body-to-body connection. Based on such differential-algebraic equations, there exist numerous integration algorithms for open-loop and/or closed-loop dynamic simulations, including those given in Refs. 1 and 2. The algorithms in Refs. 1 and 2 are particularly simple because of the development of a modified null-space method using a natural partitioning scheme. This simplicity is made possible by determining a closed-form null-space matrix in an explicit way to nullify the Jacobian matrix so that the resulting equations of motion appear to be purely differential equations only in terms of independent variables. This approach has been proven very useful and effective in dealing with various types of the MBD systems.

For linear control design for slewing systems, Ghaemmaghami and Juang[3] and Huang et al.[4] derived an alternative state transformation matrix for the Lagrangian equation of motion to localize the nonlinearities, existing in the mass matrix, into a small submatrix associated with the rigid body. It has been shown in Refs. 3 and 4 that this characteristic leads to considerable savings in computer time during simulation and design phases. The purpose of the present Note is to introduce a new alternative variable transformation incorporating a null-space transformation for the MBD system, in which the mass matrix is transformed to a band matrix[5,6] which has many numerical advantages. Such a concept forms the basis of investigation in this Note and is also demonstrated as applied to an articulated MBD system.

II. Equations of Motion of an Articulated Multibody Dynamic System

We begin to consider an MBD system featuring an open-chain configuration of n bodies articulated by the spherical joints in the presence of external forces. The equations of motion of this articulated MBD system can be represented in matrix form as

$$\boldsymbol{M}\ddot{\boldsymbol{\xi}} + \boldsymbol{B}^T\lambda = \boldsymbol{F} \quad \text{and} \quad \boldsymbol{\Phi} = \boldsymbol{B}\dot{\boldsymbol{\xi}} = 0 \tag{1}$$

where $\boldsymbol{M}$ and $\boldsymbol{B}$ denote the mass and constraint Jacobian matrices, and λ, $\boldsymbol{F}$, and $\boldsymbol{\Phi}$ the Lagrange's multiplier, forcing, and nonholonomic constraint vectors, respectively. The variable $\ddot{\boldsymbol{\xi}}$ contains the linear and angular acceleration of each body in a vectorial fashion as $[\ddot{r}_1, \dot{\omega}_1, \ldots, \ddot{r}_n, \dot{\omega}_n]^T$, and the forcing vector includes the external forces as well as the control inputs when in a closed-loop system. In expanded form, the mass matrix can be written as follows:

$$\boldsymbol{M} = \text{Diag}[\boldsymbol{m}_1, \boldsymbol{j}_1, \ldots, \boldsymbol{m}_n, \boldsymbol{j}_n] \tag{2}$$

where $\boldsymbol{j}_i$ is a submatrix containing the moments of inertia for the ith body, and $\boldsymbol{m}_i$ is a component mass matrix.

Received May 26, 1993; revision received Sept. 10, 1993; accepted for publication Sept. 10, 1993.

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